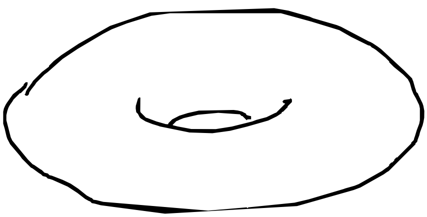
Introduction to Topology (MA 564)



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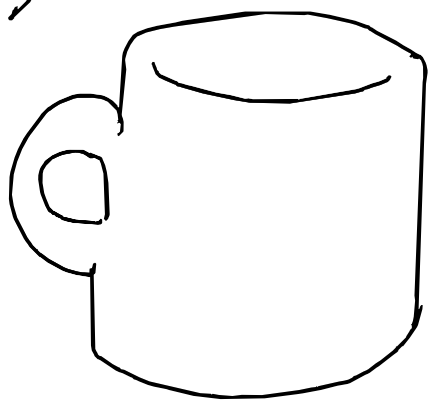


All information will be posted at

https://malavikamukundan.github.io/564.html

Please submit homework on Blackboard

REVIEW OF SET THEORY

Introduction

X : a **set**, a collection of objects.

Examples: $X = \{1,2,3\}$, $\{\}$, $\{\{\}\}$, $\{\text{apple}, \pi\}$

$a \in X$: The **element** a belongs to X
$a \notin X$: " a does not belong to X

$X \subseteq Y$: If $a \in X$, then $a \in Y$ [if not true, write $X \nsubseteq Y$]

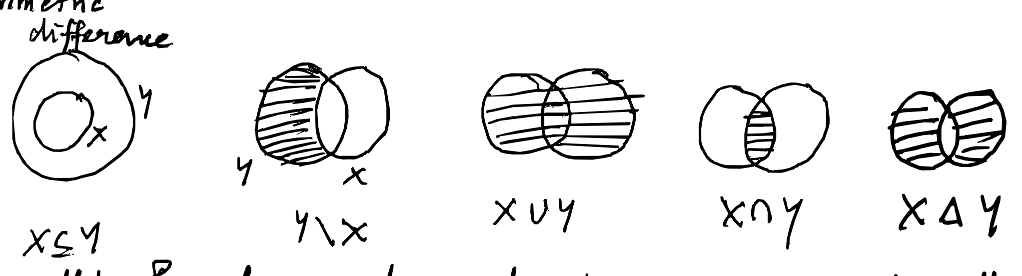
$Y \setminus X = \{a \in Y : a \notin X\}$

$X \cup Y = \{a : a \in X \text{ or } a \in Y\}$
union

$X \cap Y = \{a : a \in X \text{ and } a \in Y\}$
intersection

[Given a set I, and a collection of sets $\{X_i\}_{i \in I}$, $\bigcup_{i \in I} X_i = \{a : a \in X_i \text{ for some } i \in I\}$]

$X \Delta Y = \{a : a \in X \setminus Y \text{ or } a \in Y \setminus X\} = (X \setminus Y) \cup (Y \setminus X)$
symmetric difference



$X \subseteq Y$ $\quad$ $Y \setminus X$ $\quad$ $X \cup Y$ $\quad$ $X \cap Y$ $\quad$ $X \Delta Y$

Russell's Paradox: Let $\mathcal{A}$ be the set of all sets.
Let $X \subseteq \mathcal{A}$ the set of all sets Y s.t. $Y \notin Y$.
Neither $X \in X$ nor $X \notin X$ can hold.

Solution: $\mathcal{A}$ is not a set.
"Not every collection is a set".

Common sets of numbers: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$

Relations and Functions

Let X, Y be sets.

$$X \times Y = \{(a,b) : a \in X \text{ and } b \in Y\}.$$

Any subset R of $X \times X$ is called a relation on X

Eg: For $X = \mathbb{N} = \{1,2,3,4,\ldots\}$

$$R = \bigcup_{m \in \mathbb{N}} \{(m,n) : n \in \mathbb{N} \setminus \{1,2,\ldots,m-1\}\} = \{(m,n) : m \leq n\}$$

An "equivalence" relation $R \subseteq X \times X$ is one which is
(i) reflexive: $(a,a) \in R \;\forall a \in X$
(ii) symmetric: $(a,b) \in R \Rightarrow (b,a) \in R$
(iii) transitive: $(a,b), (b,c) \in R \Rightarrow (a,c) \in R$

Let $[a] = \{b : (a,b) \in R\}$. Then $X/R = \{[a] : a \in X\}$.
"equivalence class" of a

Eg: $R = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$ is an equivalence relation on $\{a,b,c\} = X$
$[a] = \{a\}$, $[b] = [c] = \{b,c\}$.
So $X/R = \{[a], [b]\}$.

A **function** ("map") $f : X \longrightarrow Y$ ("domain", "codomain") is a subset C of $X \times Y$ s.t. $\forall a \in X$ ("for all"), $\exists!\, b \in Y$ ("there exists a unique") s.t. $(a,b) \in C$. We call b as $f(x)$.

So $C = \{(a, f(a)) : a \in X\}$.

f is called injective/one-one/1-1 if $\forall a, b$, $f(a) = f(b) \Rightarrow a = b$

f is called surjective/onto if $f(X) = \{f(a) : a \in X\} = Y$ ("range of f")

f is called bijective/one-one correspondence if it is both one-one and onto.

The example $f(n) = n+1$ above is injective but not surjective from $\mathbb{N}$ to $\mathbb{N}$!
The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $f(n) = -n$ is bijective

Operations on functions:

Given $f : X \longrightarrow Y$, and $Z \subseteq Y$, $f^{-1}(Z) = \{a \in X : f(a) \in Z\}$.

If $f, g : X \longrightarrow \mathbb{R}$, we can define $f+g$, $f-g$, $f \cdot g$ on X and f/g on the domain $X \setminus g^{-1}(\{0\})$.

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$,
$g \circ f(x) = g(f(x))$ $\quad [g \circ f : X \longrightarrow Z]$
composition

If $f : X \longrightarrow Y$ is bijective, $f^{-1} : Y \longrightarrow X$ is $f^{-1}(b)$ is the unique element of $f^{-1}(\{b\})$.

Given f, g as above, we say g is a left inverse for f (or alternately f is a right inverse for g) if $Z \subseteq X$ and $\circ f = id_X$.

Cardinality

We say X and Y have same cardinality [$X \cong Y$]
$\exists$ a bijection $f : X \longrightarrow Y$.

X is finite if $\exists$ an injection $f : X \longrightarrow \mathbb{N}$ ["$f : X \hookrightarrow \mathbb{N}$"]
infinite o/w.

X is countably infinite if $\exists$ a bijection $f : X \longrightarrow \mathbb{N}$
X is countable if it is either finite/countably infinite
uncountable o/w.

Eg: $\mathbb{N} \cong \mathbb{Z}$.

Proof: Consider $f : \mathbb{N} \longrightarrow \mathbb{Z}$ given by

$$f(n) = \begin{cases} \frac{n-1}{2} & : n \text{ is odd} \\ -\frac{n}{2} & : n \text{ is even} \end{cases}$$

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \;\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 \end{array}$$

f is a bijection.

Exercise: $\mathbb{Z} \cong \mathbb{Q}$.

$X^Y = \{f : X \longrightarrow Y\}$.

By this definition, $\{0,1\}^Y = \{f : \{0,1\} \longrightarrow Y\}$.
We will denote $\{0,1\}^Y$ by 2^Y

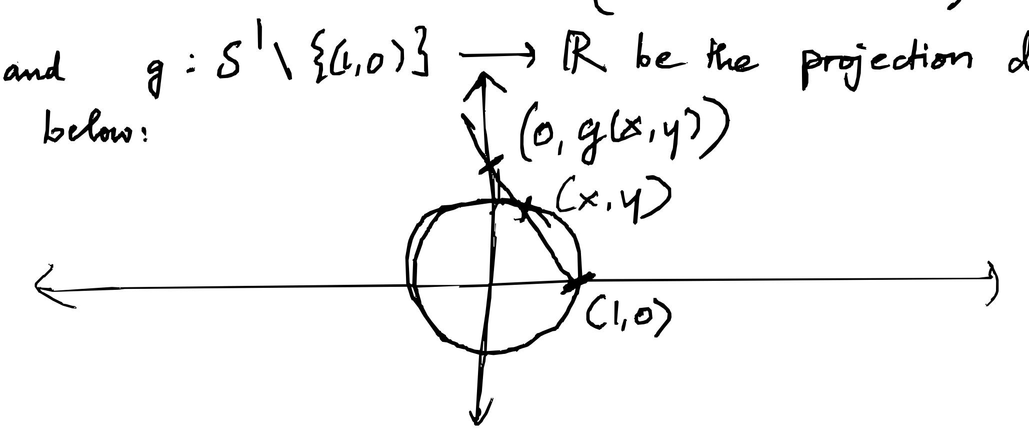
$P(X) = \{Y : Y \subseteq X\}$ $\quad$ [Note: $2^X \cong P(X)$]
power set of X

Eg: $P(\{1,2\}) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$.

$(0,1) \cong \mathbb{R}$
$\{a \in \mathbb{R} : 0 < a < 1\}$

Consider $f : (0,1) \rightarrow S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ given by
$a \longmapsto e^{2\pi i a} = (\cos(2\pi i a), \sin(2\pi a))$

and $g : S^1 \setminus \{(1,0)\} \longrightarrow \mathbb{R}$ be the projection described below:



The map $g \circ f : (0,1) \longrightarrow \mathbb{R}$ is a bijection, since
(exercise) (i) $f : (0,1) \longrightarrow S^1 \setminus \{(1,0)\}$ is a bijection
(ii) g is a bijection

Theorem: $\forall$ sets X, $X \ncong P(X)$.
Theorem: $2^{\mathbb{N}} \cong \mathbb{R}$

[see Munkres, Chapter 1]

Axiom of Choice

Let $\mathcal{A}$ be a collection of disjoint non-empty sets. Then there exists a set C consisting of exactly one element from each $A \in \mathcal{A}$.

Application (eg): Every vector space has a basis.